Doctoral School of Finance and Banking

Multifractality in interest rate modelling

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Introduction

- Two major discrepancies between the Bachelier model and actual financial data:
  - Temporal dependence in price changes
  - Much fatter tails than the Gaussian distribution
- Mandelbrot, Calvet and Fisher (1997) introduce the concept of multifractality to economics through the Multifractal Model of Asset Returns.
- The model incorporates long-tailed asset returns, long memory in volatility and scale-consistency of financial series.

Multifractality

- Multifractal measures: Mandelbrot (1972)
  - Multiplicative cascades
- Multifractal processes: Mandelbrot, Calvet and Fisher (1997)
- Multifractality is defined as a global property of the process' moments:

$$\mathrm{E}\left(\left|X(t)\right|^{q}\right)=c(q)t^{\tau(q)+1}$$

- The function τ(q) is called the scaling function of the multifractal process.

The Multifractal Model of Asset Returns

- Mandelbrot, Calvet and Fisher (1997)
- Considering the price of a financial asset P(t) on [0, T], the following stochastic process is defined:

$$X(t) = \ln P(t) - \ln P(0)$$

- Assumptions:
  1. A compound process between a fractional Brownian Motion and a stochastic trading time

$$X(t) \equiv B_H\left[\theta(t)\right]$$

  2. Trading time is the CDF of a multifractal measure on [0, T]
  3. The fractional Brownian Motion and trading time are independent
- Under Assumptions 1-3, the process X(t) is multifractal, with stationary increments.

Methodology

- Straightforward and practical procedure
  - Detection of time and frequency scaling properties
  - Estimation of particular parameters
  - Synthesis of the multifractal measure
  - Simulation of the compound process

Partition function

- A simple test of multifractality
- Dividing [0,T] into N intervals of length δ, the partition function is defined as:

$$S_\delta(T,q)=\sum_{i=1}^{N}\left|X_{\lceil i\cdot\delta\rceil}-X_{\lceil (i-1)\cdot\delta\rceil}\right|^q$$

- If X(t) is multifractal, the scaling law yields:

$$\log\big(S_\delta(T,q)\big)\approx\tau(q)\log(\delta)+\log[c(q)]+\log(T)$$

  when the q^{th} moment exists.

  - The linearity of the partition function plots for given values of q is proposed as a test of the MMAR.

Scaling function

- The slope of the partition function gives an estimate of the scaling function $\tau(q)$.
- Using various moments q and incremental time δ, the scaling function can be identified.
- Mandelbrot, Calvet and Fisher (1997) show that:
  - $\tau(0) = -1$
  - $\tau(q)$ is concave for multiscaling processes and linear for uniscaling processes
  - $\tau_X(q) \equiv \tau_\theta(Hq)$
  - $\tau_X(1/H) = \tau_\theta(1) = 0$

Hölder exponents

- Mandelbrot, Calvet and Fisher (1997) consider the infinitesimal variation in price around a date t as:

$$\left|P(t+dt)-P(t)\right| \sim \mathrm{C}_{\mathrm{t}}\left(dt\right)^{\alpha(t)}$$

  where α(t) is the local Hölder exponent at t.
- The local Hurst exponent appears as a local scale.
- The MMAR contains a continuum of local Hölder exponents, contrary to previous models that contain at most two Hölder exponents along their sample paths.

Multifractal spectrum

- Describes the distribution of Hölder exponents in a multifractal process.
- The multifractal spectrum f(α) is the Legendre transform of the scaling function τ(q):

$$f(\alpha) = \underset{q}{Min}\left[\alpha q - \tau(q)\right]$$

- Mandelbrot, Calvet and Fisher (1997) show that:
  - $f_X(\alpha) \equiv f_\theta(\alpha/H)$

Multifractal measure

- Generated by a lognormal multiplier M with distribution:

$$\log M \sim \mathrm{N}(\lambda, \sigma^2)$$

- The first two moments of the distribution are:

$$\lambda = \frac{\alpha_0}{H} \quad \text{and} \quad \sigma^2 = \frac{2(\lambda - 1)}{\log b}$$

- The most probable exponent of trading time is:

$$\alpha_0 = \lambda H$$

- Calvet and Fisher (2002) show that the multifractal spectrum of trading time is a quadratic function:

$$f_\theta(\alpha) = 1 - \frac{(\alpha - \lambda)^2}{4(\lambda - 1)}$$

Simulation of the MMAR

- Generate the multifractal probability measure with k steps, such that $2^k \geq T$. The cumulative sum of measures represents trading time.
- Generate a discretized path of a fBM, by simulating the first differences of the fBM, process known as the fractional Gaussian Noise. The covariance of the fGN at a lag k is

$$\gamma(k) = \frac{\sigma^2}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right)$$

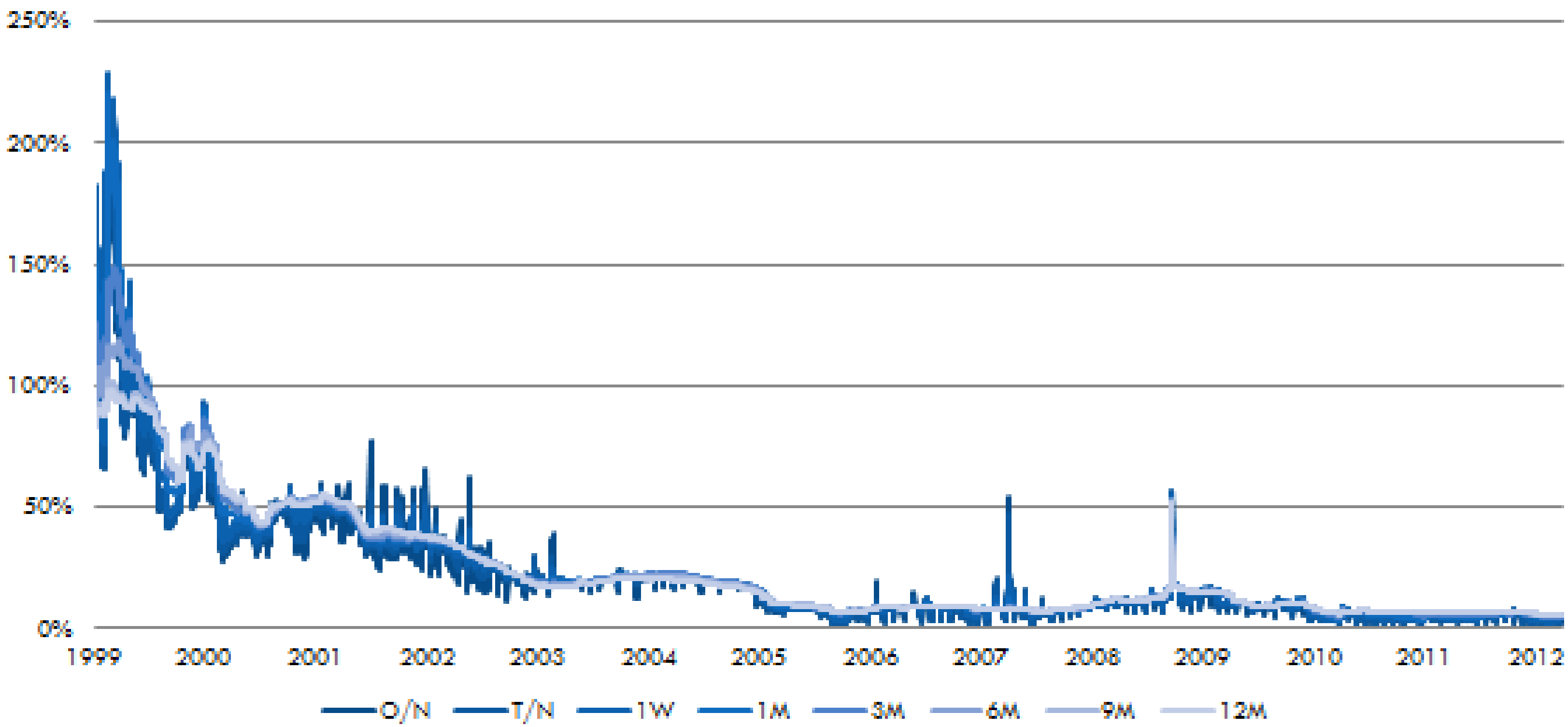
- The simulated fBM is compounded with trading time using interpolation.

Data

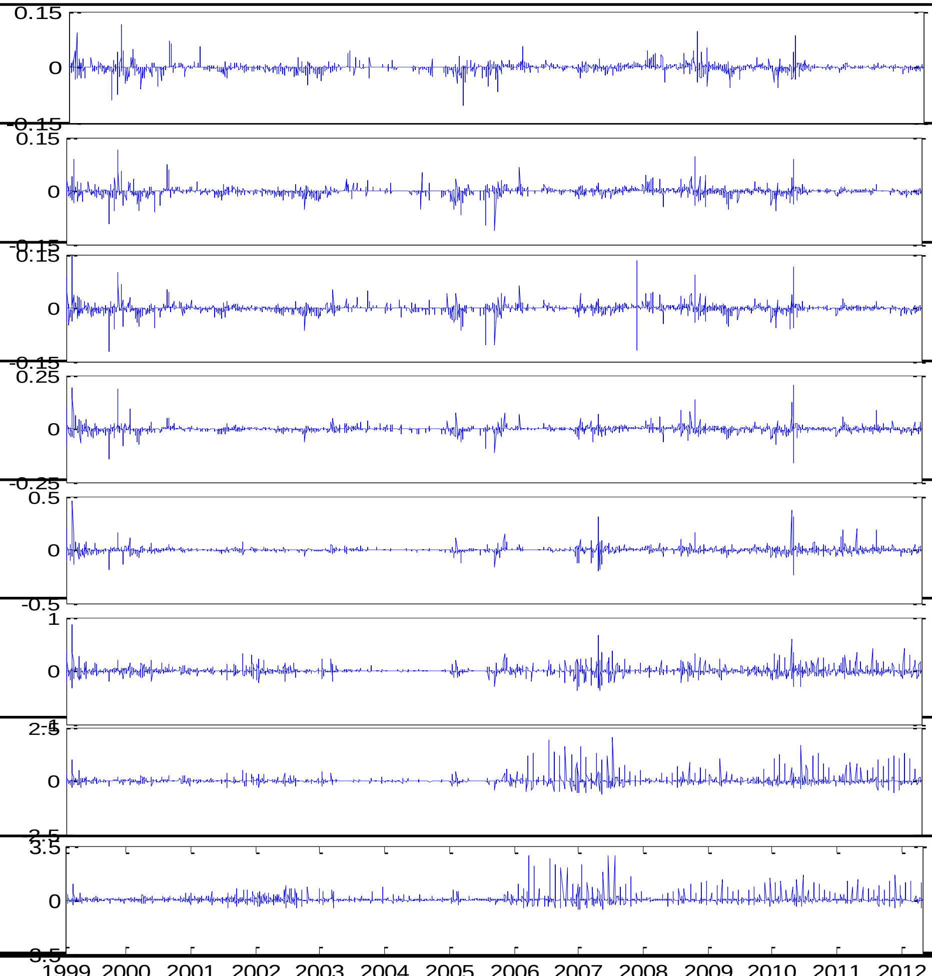
- Romanian Interbank Offered Rates (ROBOR), with overnight (O/N), tomorrow next (T/N), 1-week (1W), 1-month (1M), 3-month (3M), 6-month (6M), 9-month (9M) and 12-month (12M) maturities.
- Source: National Bank of Romania
- Period: February 04, 1999 to April 26, 2012
- 3372 daily observations

ROBOR series

- Graphic representation of the 8 studied ROBOR series

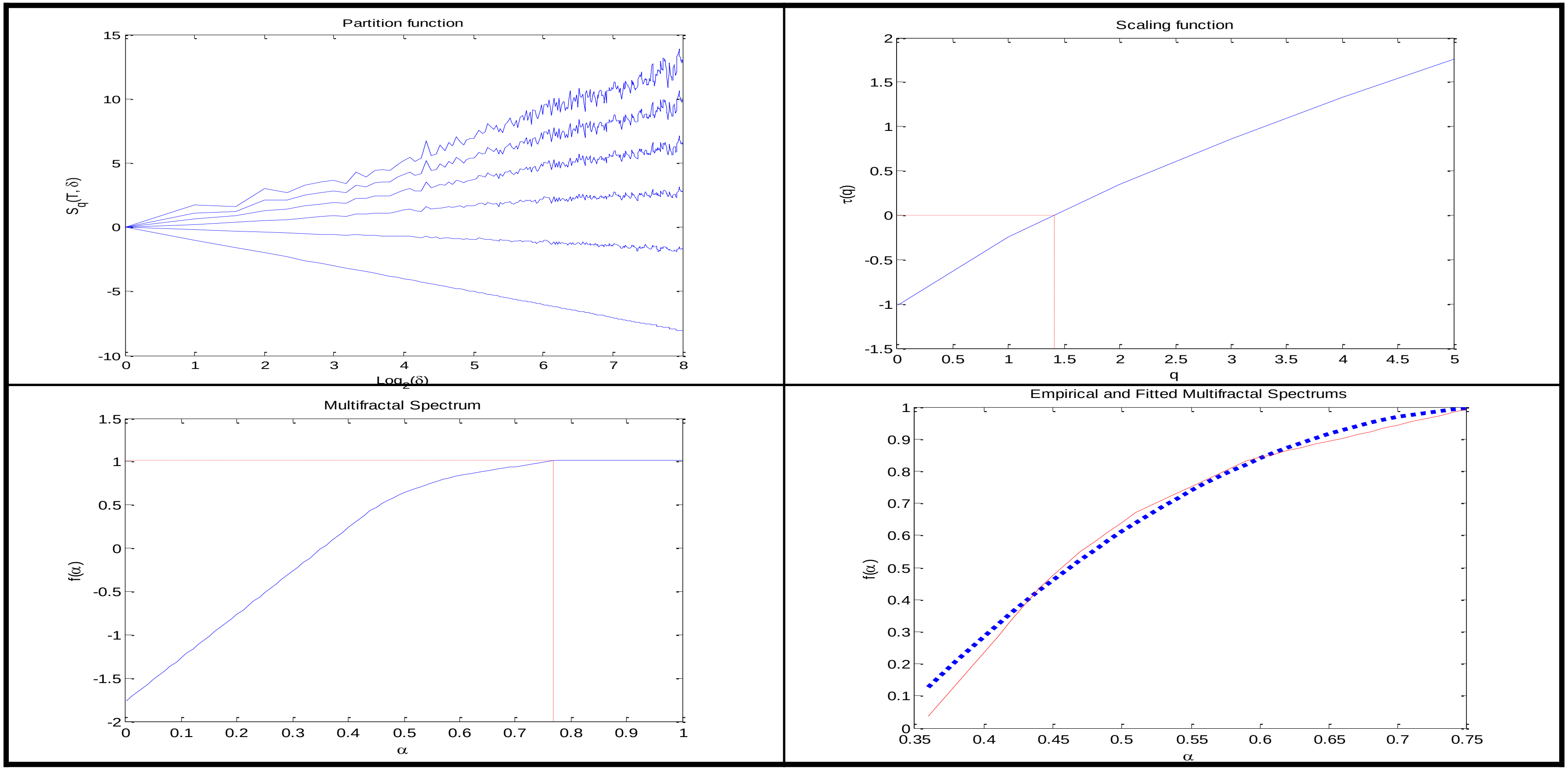


Return series

Maturity		H	α_0
12M		0.71	0.76
9M		0.71	0.76
6M		0.69	0.74
3M		0.62	0.68
1M		0.48	0.59
1W		0.26	0.41
TN		0.24	0.58
ON		0.27	0.60

Time-scale analytical profile

- ROBOR 12-month



MMAR parameters

Maturity	Pricing Hurst (H)	Trading Time Hurst (α_0)	Mean (λ)	Variance (σ^2)
12M	0.7087	0.7600	1.0723	0.1474
9M	0.7118	0.7600	1.0677	0.1353
6M	0.6905	0.7400	1.0717	0.1433
3M	0.6186	0.6800	1.0993	0.1986
1M	0.4813	0.5900	1.2259	0.4517
1W	0.2604	0.4100	1.5744	1.1488
TN	0.2372	0.5800	2.4454	2.8908
ON	0.2707	0.6000	2.2161	2.4323

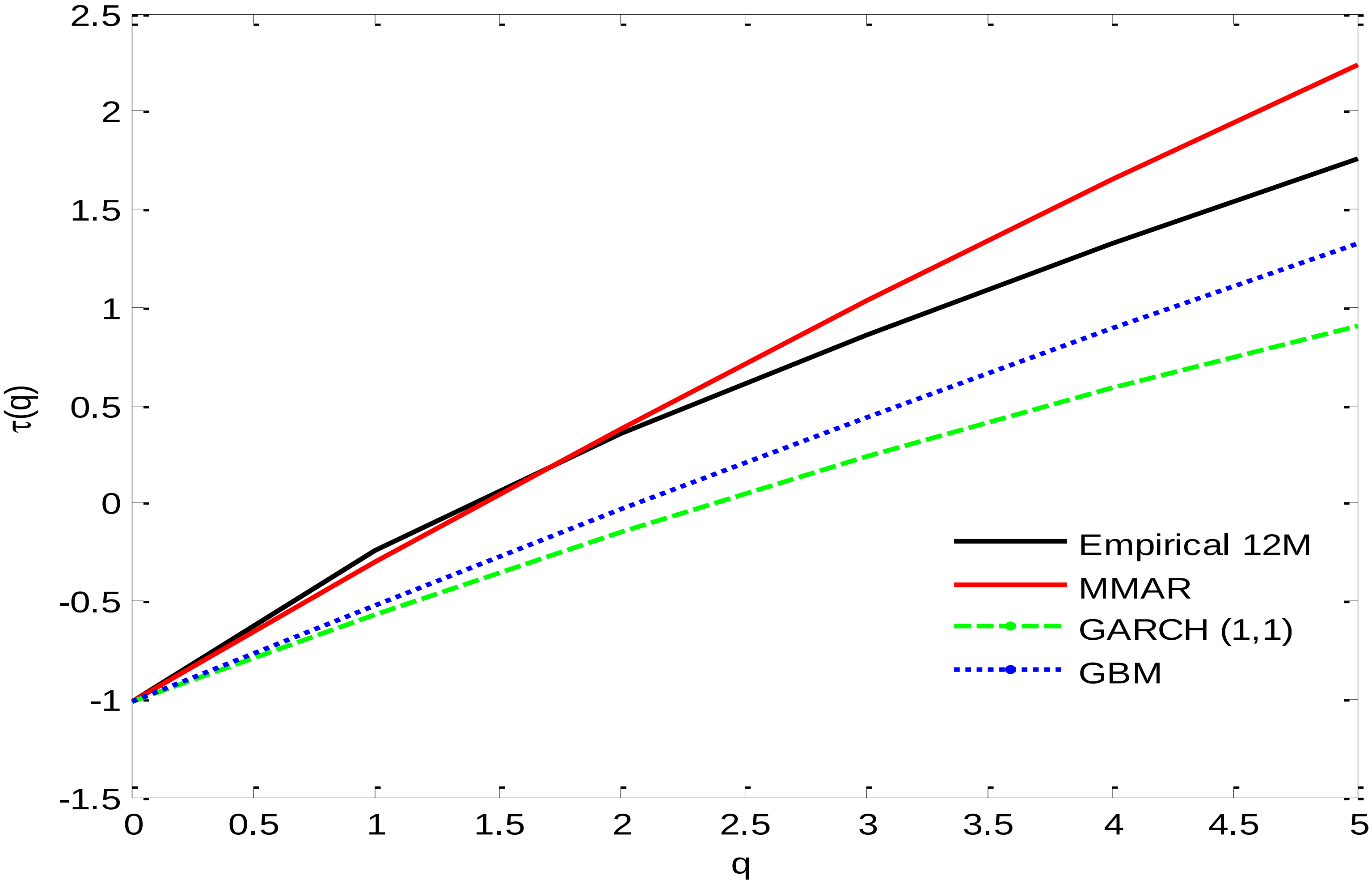
Long memory

Maturity	Averaged Simulated Hurst	Empirical Hurst
12M	0.70	0.71
9M	0.70	0.71
6M	0.68	0.69
3M	0.61	0.62
1M	0.46	0.48
1W	0.33	0.26
TN	0.35	0.24
ON	0.36	0.27

Scaling properties

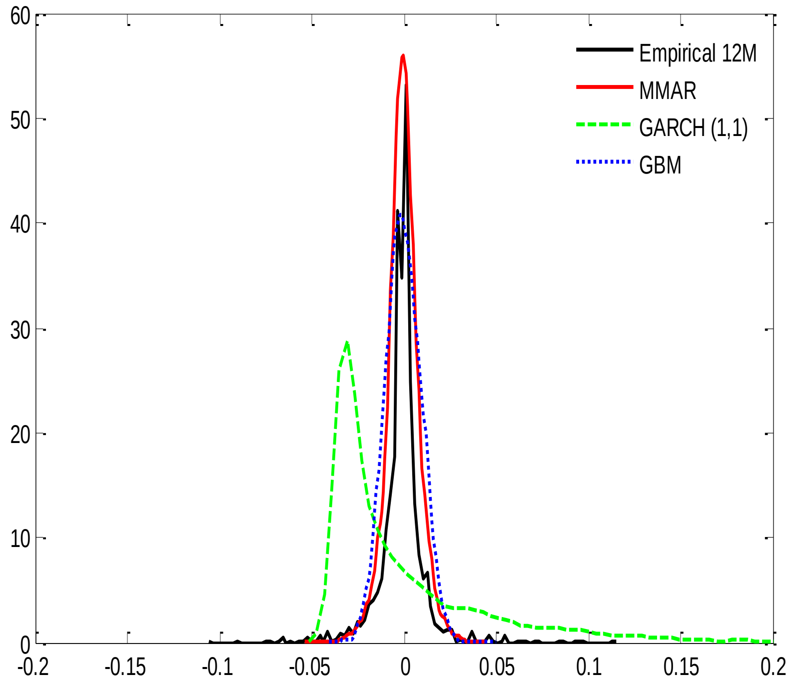
Moment (q)	Empirical series	Simulated MMAR	Simulated GARCH (1,1)	Simulated GBM
0	-1.01	-1.01	-1.01	-1.01
1	-0.24	-0.30	-0.57	-0.52
2	0.35	0.38	-0.15	-0.03
3	0.86	1.03	0.24	0.44
4	1.33	1.65	0.59	0.89
5	1.76	2.24	0.90	1.33

Scaling functions comparison

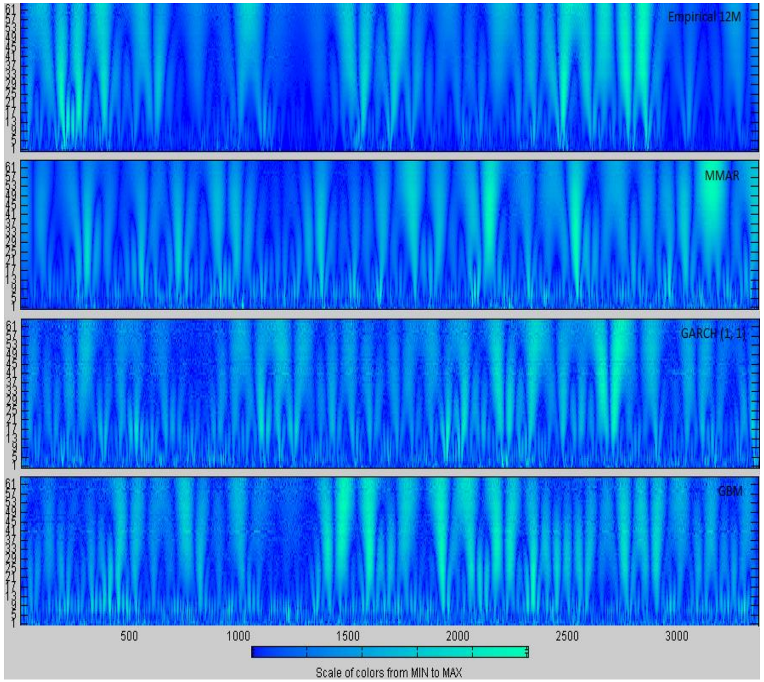


Simulation comparison

- Density estimation



- Scalogram



Conclusions

- The paper shows that the ROBOR series are multifractal processes.
- Straight and smooth partition functions for an investment horizon up to one business year provide evidence that the scaling rule exists for at least five moments.
- However, for short-term maturities, such as tomorrow-next and overnight, the partition functions collapse for longer horizons.
- The curvature of the scaling functions supports multifractality for all series.
- Hurst exponents are higher than 0.5 for maturities longer than 3 months, meaning that the respective series have long memory.
- The MMAR preserves long memory, but it has some difficulties with anti-persistent series.
- For maturities longer than 1 month, long tails and high peaks are well preserved by the MMAR simulation.
- The MMAR can be adapted to single-asset volatility models, since the paper proved that the model can trace the second moment of the empirical series and given the existence of long memory in the volatility of financial time series.

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