Stochastic volatility model under Bayesian Approach

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Presentation structure

Purpose of the paper

Literature review

Stochastic volatility models

Estimation methods

Data description

Results and interpretations

Conclusion and references

Purpose of the paper

- To calibrate stochastic volatility models with different characteristics using Bayesian inference techniques

- To analyze in a comparative manner the predictive ability of the models

- To perform empirical computations for four blue-chips indices from CEE capital markets: BET (Romania), PX (Czech Republic), WIG20 (Poland), ATX (Austria).

- To draw conclusions based on results

Volatility in economics

Definition: "The extent to which the price of a security or commodity, or the level of a market, interest rate or currency, changes over time. High volatility implies rapid and large upward and downward movements over a relatively short period of time; low volatility implies much smaller and less frequent changes in value." (Neil Shepard)

General volatility measures

- **Standard deviation**
- **Beta**

General approach:
According to "Portfolio Selection Theory" (Markowitz) and CAPM (Sharpe), higher expected returns can only occur with correspondingly higher risk.

Importance of volatility forecasting

An accurate forecast of volatility is essential to:

- **optimal asset allocation**
- **derivatives pricing**
- **dynamic hedging**
- **portfolio risk management**

Stochastic volatility models

Neil Shepard: "Stochastic volatility (SV) is the main concept used in the fields of financial economics and mathematical finance to deal with the endemic time-varying volatility and codependence found in financial markets."

SV => one approach to resolve a shortcoming of the Black-Scholes model

SV preset the ability of the stochastic volatility models to capture some important properties of financial time series:

- Volatility clustering and persistence
- Thick tails
- Leverage effect
- Extreme events in returns

Stochastic volatility model

SV model is often formulated in terms of s. d. e.:

$$ds(t) = \sigma(t) * dB_1(t)$$

$$d\ln\sigma^2(t) = \alpha + \beta * ln\,\sigma^2(t) * dt + \eta * dB_2(t)$$

$$corr\big(dB_1(t),\ dB_2(t)\big) = 0$$

Using the notations ➕ Euler-Maruyama Approx. ➡ Discrete SV model

$$s(t+1) - s(t) = y(t)$$

$$B_1(t+1) - B_1(t) = u_t$$

$$B_2(t+1) - B_2(t) = \eta_t$$

$$1 + \beta = \phi$$

$$ln\,\sigma^2(t) = h_t$$

$$\mu = \alpha\,(1 + \phi)$$

➡

$$y_t = \sigma_t\, u_t = \exp(h_t / 2)\, u_t$$

$$h_{t+1} = \mu + \phi * (h_t - \mu) + \sigma_\eta \eta_{t+1}$$

$$u_t \sim N(0,1)\ i.i.d.,\ \eta_t \sim N(0,1)\ i.i.d.$$

Discrete stochastic volatility models

SV

$$y_t|h_t = \sigma_t\, u_t \;=\; \exp\left(\frac{h_t}{2}\right) * \; u_t \quad , \qquad u_t \sim N(0,1)\; i.i.d.$$

$$h_{t+1}|h_t,\mu,\phi,\sigma_\eta = \;\mu + \;\phi * (\, h_t - \;\mu\,) + \sigma_\eta \eta_{t+1}\, , \qquad \eta_t \sim N(0,1)\; i.i.d.$$

$$t = 1, \dots, T \qquad h_1 \sim N\left(\mu\,, \frac{\sigma_\eta^2}{1-\;\phi^2}\right) \qquad corr(\,u_t\,, \eta_t\,) = \;0 \qquad |\,\phi\,| \;<\; 1$$

SV $\Rightarrow$ Volatility clustering + Leptokurtic distribution

$\not\Rightarrow$ Excess kurtosis

To characterize the “Excess kurtosis”, the SVt model is defined:

SVt

$$u_t \sim t_k\; i.i.d., \qquad t = 1, \dots, T$$

Discrete stochastic volatility models

Leverage effect → changes in the asset prices tend to be negatively correlated with changes in volatility of assets

The standard SV model is extended and SVL is obtained:

SVL

$$y_t | h_t, \rho = \exp(h_t / 2) * u_t \qquad t = 1, \dots, T$$

$$h_{t+1} | h_t, \mu, \phi, \sigma_\eta = \mu + \phi * (h_t - \mu) + \sigma_\eta \eta_{t+1}$$

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

Discrete stochastic volatility models

Financial crisis (2008)
European debt crisis (2011)

→ importance of modeling **extreme events** in the financial time series

Jump Component

Continuous component

$$y_t = k_t q_t \quad + \quad \exp(h_t \,/\, 2\,)\, u_t$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \eta_t$$

SVLJ

$$k_t \sim N(\alpha_k\,, \beta_k) \qquad q_t \sim Bern(\kappa)$$

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\,\sigma_\eta \\ \rho\,\sigma_\eta & \sigma_\eta \end{pmatrix} \right)$$

Estimation method

Priors Likelihood

Joint probability distribution

$$p(y,\theta) = p(\theta) * p(y \mid \theta) = p(y) * p(\theta \mid y)$$

Bayes Theorem

$$y = (y_1, \dots, y_n) \qquad \theta = (\mu, \phi, \sigma_\eta, \rho, k, h)$$

Joint posterior distribution

$$p(\theta \mid y) = \frac{p(\theta) * p(y \mid \theta)}{\int p(\theta) * p(y \mid \theta)\, d\theta} \propto p(\theta) * p(y \mid \theta)$$

SV

$$p\left(\mu, \phi, \sigma_\eta^2, h_1, \dots, h_n \mid y_1, \dots, y_n\right) \propto p(\mu)\, p(\phi)\, p\left(\sigma_\eta^2\right) \prod_{t=1}^{n} p(h_t \mid h_{t-1}, \mu, \phi, \sigma_\eta^2) \prod_{t=1}^{n} p(y_t \mid h_t)$$

$$p\left(h_t \mid h_{t-1}, \mu, \phi, \sigma_\eta^2\right) \Rightarrow N(\mu + h_{t-1}, \sigma_\eta^2)$$

$$p(y_t \mid h_t) \Rightarrow N(0, \exp(h_t))$$

Estimation method

Joint posterior distribution **SVt**

Likelihood $p(y_t | h_t) \Rightarrow t(0, \exp(h_t), k)$

Prior distribution for ***k***, the **degrees of freedom**.

Joint posterior distribution **SVL**

$$y_t \sim N\left(\left(\frac{\rho}{\sigma_\eta}\right)\exp\left(\frac{h_t}{2}\right)(h_{t+1} - \mu - \phi(h_t - \mu)), \exp(h_t) * (1 - \rho^2)\right)$$

$$h_{t+1} \sim N(\mu + \phi(h_t - \mu), {\sigma_\eta}^2)$$

(According to Harvey and Shepard)

Apriori distributions

$$\mu \sim N(0, 25)$$

$$\phi^* \sim Beta(20, 1.5), \phi = 2\phi - 1$$

$${\sigma_\eta}^2 \sim InverseGamma(2.5, 0.025)$$

$$\rho \sim Uniform(-1,1)$$

Estimation method

Not possible to derive an analytic expression for the posterior distribution of parameters and latent states → Gibbs Sampling

- Initialize $h^{(0)}, \mu^{(0)}, \phi^{(0)}$ and $\sigma_\eta^{2\,(0)}$
- For $g = 1, \ldots, G$:
    - For $t = 1, \ldots, T$:

      Sample $h_t^{(g)}$ from $p(h_t \,|y, h_{<t}^{(g)}, h_{>t}^{(g-1)}, \mu^{(g-1)}, \phi^{(g-1)}$ and $\sigma_\eta^{2\,(g-1)})$.
    - Sample $\sigma_\eta^{2\,(g)}$ from $p(\,\sigma_\eta^2 \,|y, h^{(g)},\ \phi^{(g-1)})$
    - Sample $\phi^{(g)}$ from $p(\,\phi \,|y, h^{(g)},\ \sigma_\eta^{2\,(g)}, \mu^{(g-1)})$
    - Sample $\mu^{(g)}$ from $p(\,\mu \,|y, h^{(g)},\ \phi^{(g)},\ \sigma_\eta^{2\,(g)})$

Data description

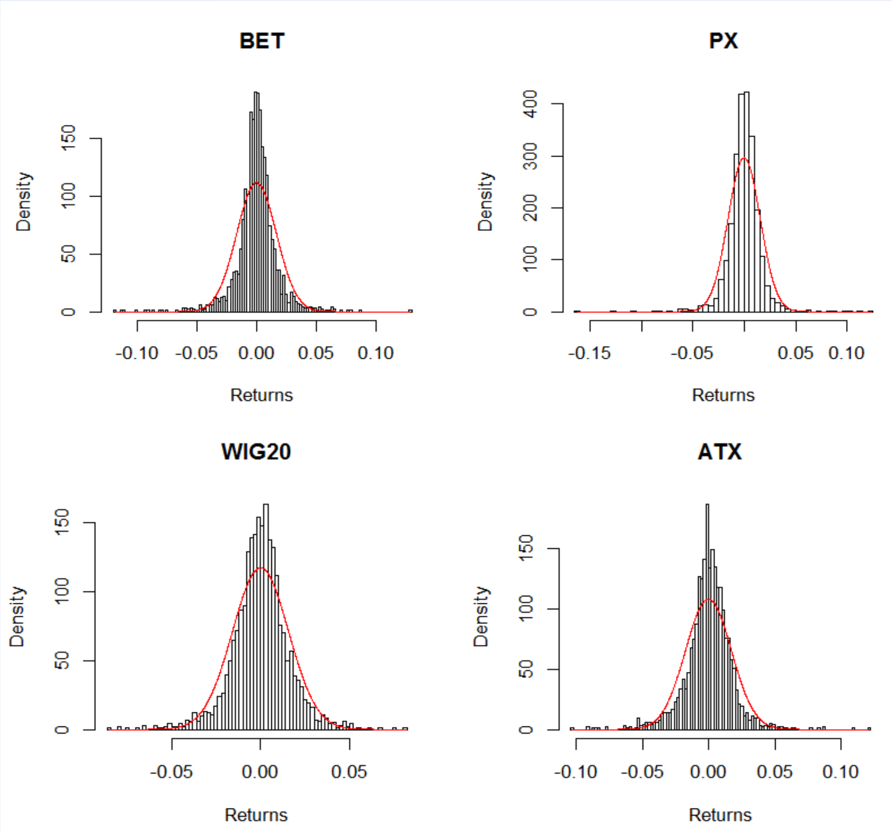
y_t = logarithmic returns of blue-chips indices from CEE stock exchanges = $\ln(S_t / S_{t-1})$

Index	Stock exchange	Stocks aggregated	Period	Observations	Source
BET	Bucharest Stock Exchange	10	Jan 3, 2006 - Mar 31, 2015	2319	Bloomberg
PX	Prague Stock Exchange	14	Jan 2, 2006 - Mar 31, 2015	2322	Bloomberg
WIG20	Warsaw Stock Exchange	20	Jan 2, 2006 - Mar 31, 2015	2314	Bloomberg
ATX	Vienna Stock Exchange	20	Jan 2, 2006 - Mar 31, 2015	2338	Bloomberg

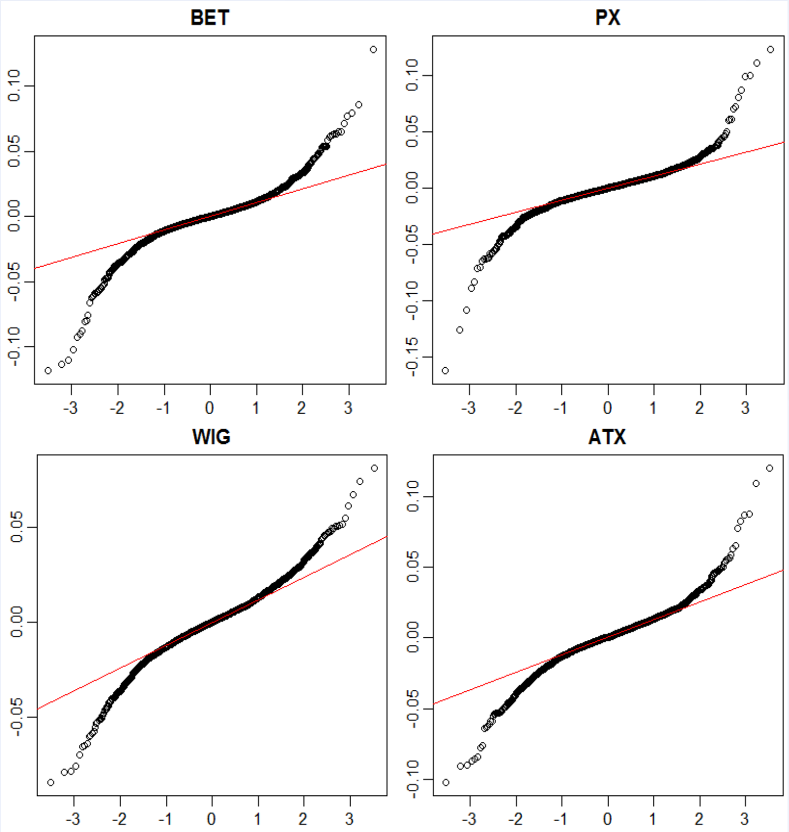
Descriptive statistics for the indices' returns time series				
Indicator	BET	PX	WIG20	ATX
Mean	1,59E-05	-0,0001497	-5,08E-05	-0,000164031
Maximum	0,1284662	0,1236405	0,08154839	0,1202104
Minimum	-0,1182396	-0,1618547	-0,08442765	-0,1025264
Std. Dev.	0,01664129	0,01563632	0,01571566	0,01724652
Skewness	-0,4628239	-0,4989881	-0,2761839	-0,2255455
Kurtosis	8,497053	14,363070	3,043631	5,374535
Jarque-Bera	7072,472	20106,03	925,3018	2840,089
P-value	0,00000	0,00000	0,00000	0,00000

Data features – Normal density test

Positive excess kurtosis confirms the usual leptokurtic distributions of stock prices' returns.



Log returns distributions vs. Normal distributions
(Source: own computes)



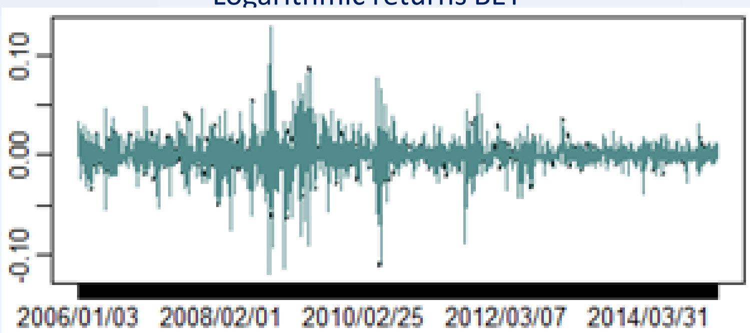
QQ plots for the log returns
(Source: own computes)

Data features – Heteroskedasticity

Time series do not present constant variance over time,
which rises the need to model the data in a time-varying framework.

Logarithmic returns BET

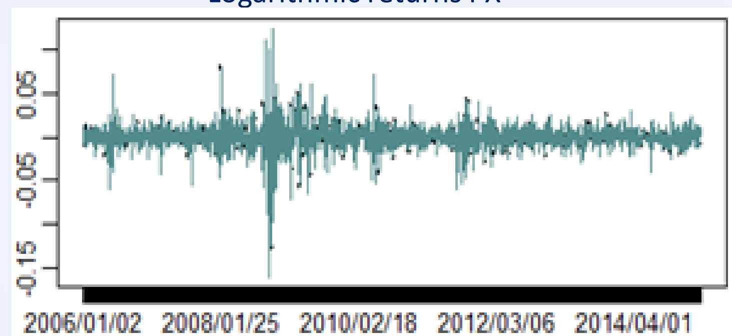
Values



Dates (yyyy/mm/dd)

Logarithmic returns PX

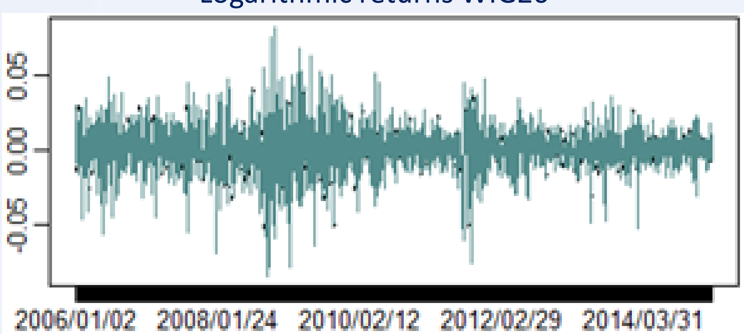
Values



Dates (yyyy/mm/dd)

Logarithmic returns WIG20

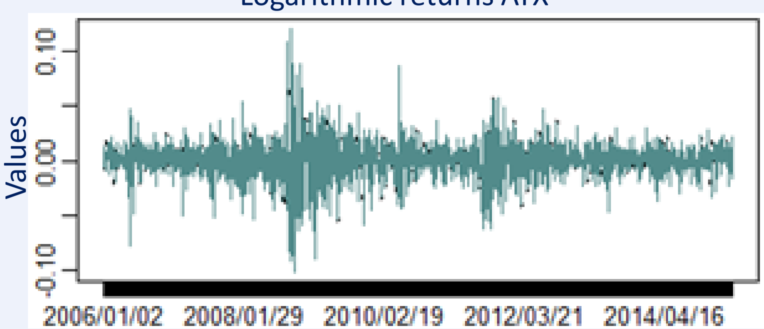
Values



Dates (yyyy/mm/dd)

Logarithmic returns ATX

Values



Dates (yyyy/mm/dd)

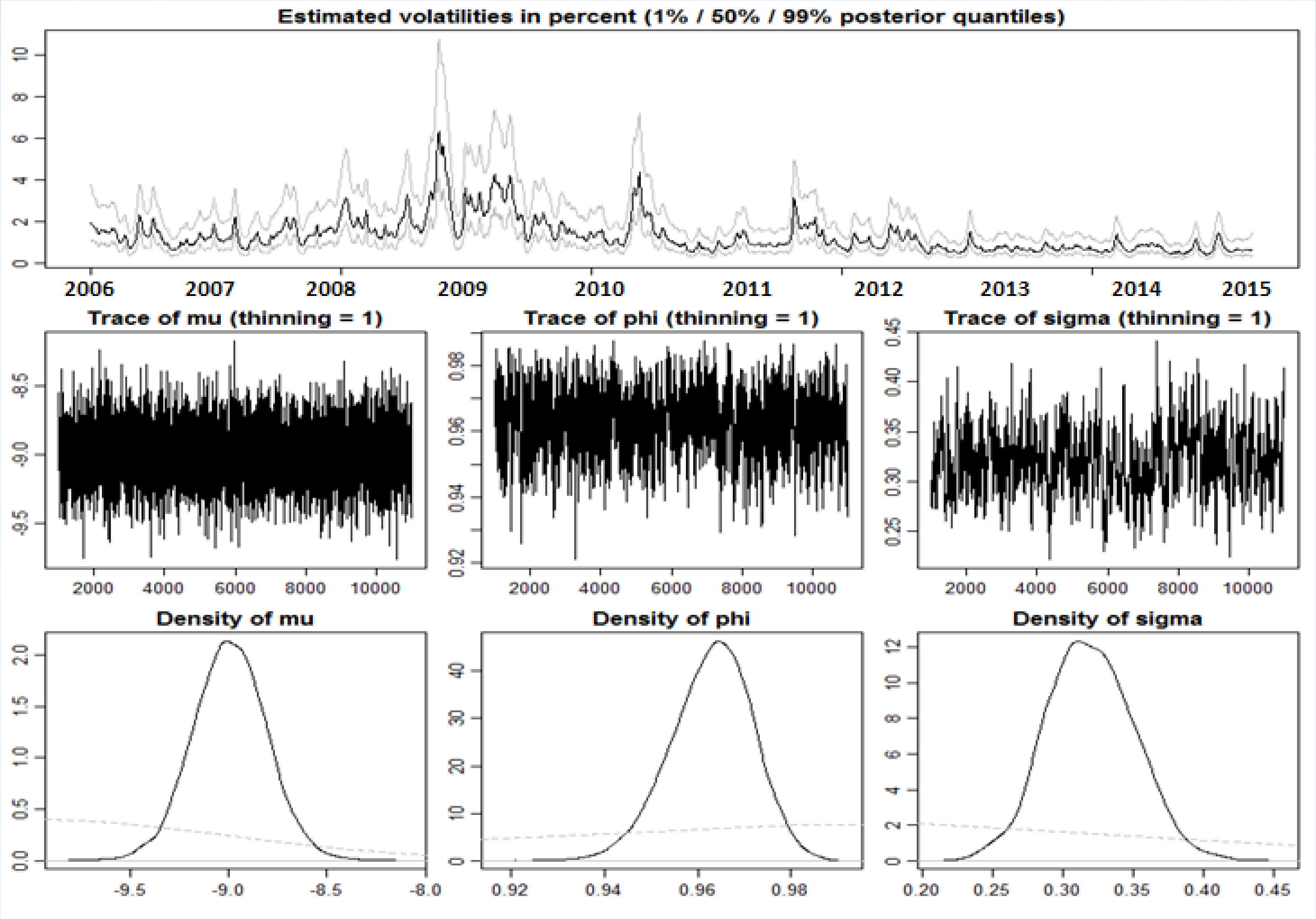
Data features – Stationarity tests

Augmented Dickey Fuller (ADF) and Kwiatokski-Phillips-Schmidts-Shin Unit Root (KPSS) tests have opposite null hypothesis, which strengthens the result that all the five data series are stationary.

	BET	PX	WIG20	ATX
ADF test	**1**	**1**	**1**	**1**
p-value	0.01	0.01	0.01	0.01
test statistic	-11.2597	-11.7924	-13.5123	-13.01
KPSS test	**0**	**0**	**0**	**0**
p-value	0.1000	0.1000	0.1000	0.1000
test statistic	0.1812	0.073	0.0599	0.0988

Results of ADF and KPSS tests - own computes (R output)

Results and interpretations

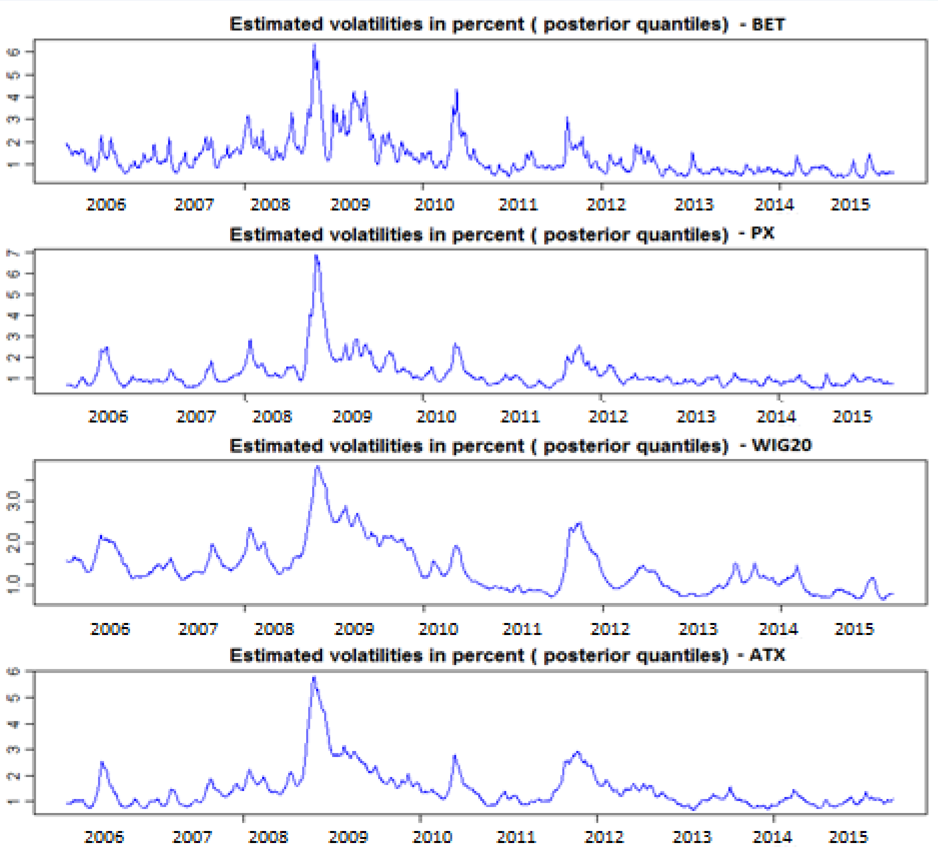


*Results of estimations for BET – **SV** model – Source: own computes (R output)*

Results and interpretations

Index	Param	Mean	SD	5%	50%	95%
BET	μ	-8.996	0.1872	-9.2997	-8.995	-8.698
	Φ	0.962	0.0087	0.9469	0.962	0.975
	σ	0.326	0.0302	0.2783	0.326	0.378
PX	μ	-9.045	0.1936	-9.3682	-9.038	-8.741
	Φ	0.976	0.0065	0.9642	0.976	0.986
	σ	0.212	0.0213	0.1796	0.210	0.249
WIG20	μ	-8.797	0.3066	-9.2917	-8.775	-8.379
	Φ	0.990	0.0040	0.9830	0.990	0.996
	σ	0.122	0.0146	0.0985	0.121	0.146
ATX	μ	-8.725	0.2601	-9.127	-8.704	-8.360
	Φ	0.984	0.0051	0.976	0.985	0.992
	σ	0.160	0.0184	0.132	0.159	0.192

*Estimated parameters – **SV** model*
(Source: own computes)

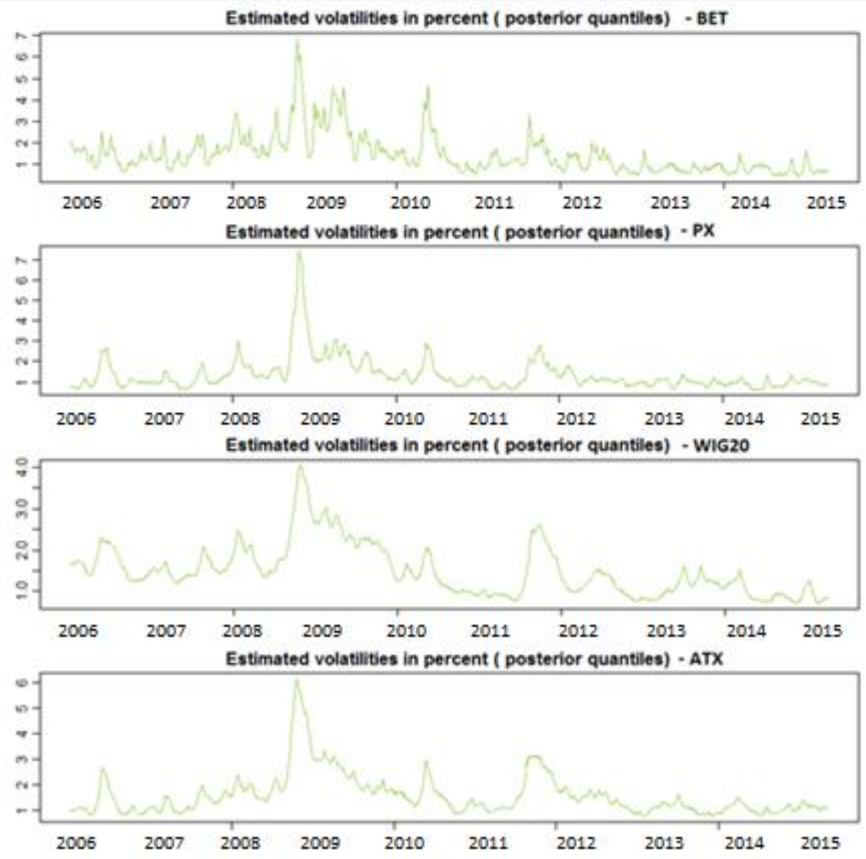


*Graphic of mean estimated volatilities – **SV** model*
(Source: own computes)

Results and interpretations

Index	Param	Mean	SD	5%	50%	95%
BET	μ	-9.126	0.2012	-9.3932	-9.125	-8.781
	Φ	0.967	0.0088	0.9431	0.967	0.981
	σ	0.329	0.0362	0.2716	0.329	0.383
	ν	13.446	1.8022	10.7002	12.446	17.871
PX	μ	-9.063	0.1947	-9.3712	-9.058	-8.781
	Φ	0.981	0.0071	0.9673	0.981	0.988
	σ	0.217	0.0214	0.1798	0.212	0.252
	ν	12.781	1.9231	9.6672	12.781	16.921
WIG20	μ	-8.823	0.3079	-9.2929	-8.824	-8.372
	Φ	0.992	0.0040	0.9832	0.992	0.996
	σ	0.123	0.0148	0.0995	0.123	0.151
	ν	14.213	1.7891	11.6022	14.213	18.873
ATX	μ	-8.726	0.2613	-9.140	-8.727	-8.374
	Φ	0.987	0.0052	0.978	0.987	0.994
	σ	0.161	0.0183	0.133	0.161	0.193
	ν	14.344	1.7923	11.6301	14.344	18.875

*Estimated parameters – **SVt** model*
(Source: own computes)

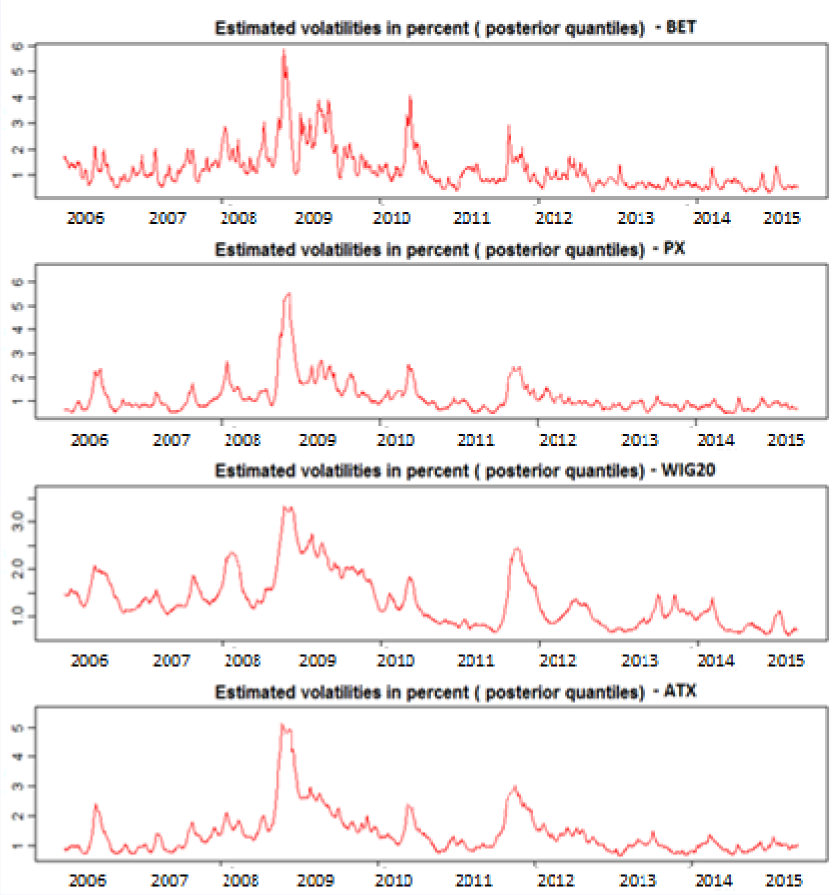


*Graphic of mean estimated volatilities – **SVt** model*
(Source: own computes)

Results and interpretations

Index	Param	Mean	SD	5%	50%	95%
BET	μ	-9.012	0.1876	-9.3075	-9.012	-8.675
	Φ	0.956	0.0067	0.9423	0.956	0.972
	σ	0.321	0.0298	0.2781	0.321	0.376
	ρ	-0.1925	0.0755	-0.3371	-0.1925	-0.0471
PX	μ	-9.062	0.2013	-9.3696	-9.061	-8.756
	Φ	0.969	0.0062	0.9637	0.968	0.979
	σ	0.213	0.0202	0.1799	0.211	0.251
	ρ	-0.2532	0.0821	-0.4012	-0.2531	-0,0721
WIG20	μ	-8.802	0.3081	-9.3012	-8.801	-8.391
	Φ	0.989	0.0042	0.9829	0.990	0.997
	σ	0.121	0.0147	0.0983	0.121	0.147
	ρ	-0.2103	0.0798	-0.3451	-0.2102	-0.052
ATX	μ	-8.743	0.2612	-9.131	-8.743	-8.365
	Φ	0.986	0.0053	0.975	0.986	0.993
	σ	0.161	0.0185	0.131	0.160	0.193
	ρ	-0.2521	0.0807	-0.3821	-0.2520	-0.064

*Estimated parameters – **SVL** model*
(Source: own computes)

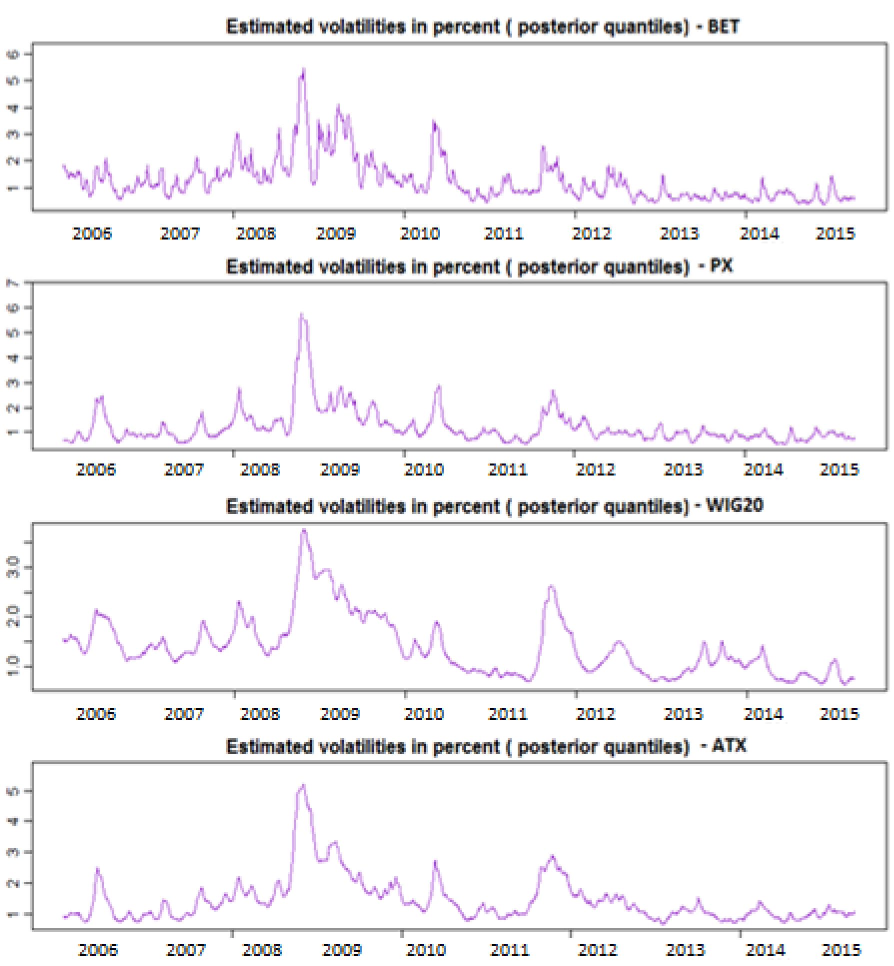


*Graphic of mean estimated volatilities – **SVL** model*
(Source: own computes)

Results and interpretations

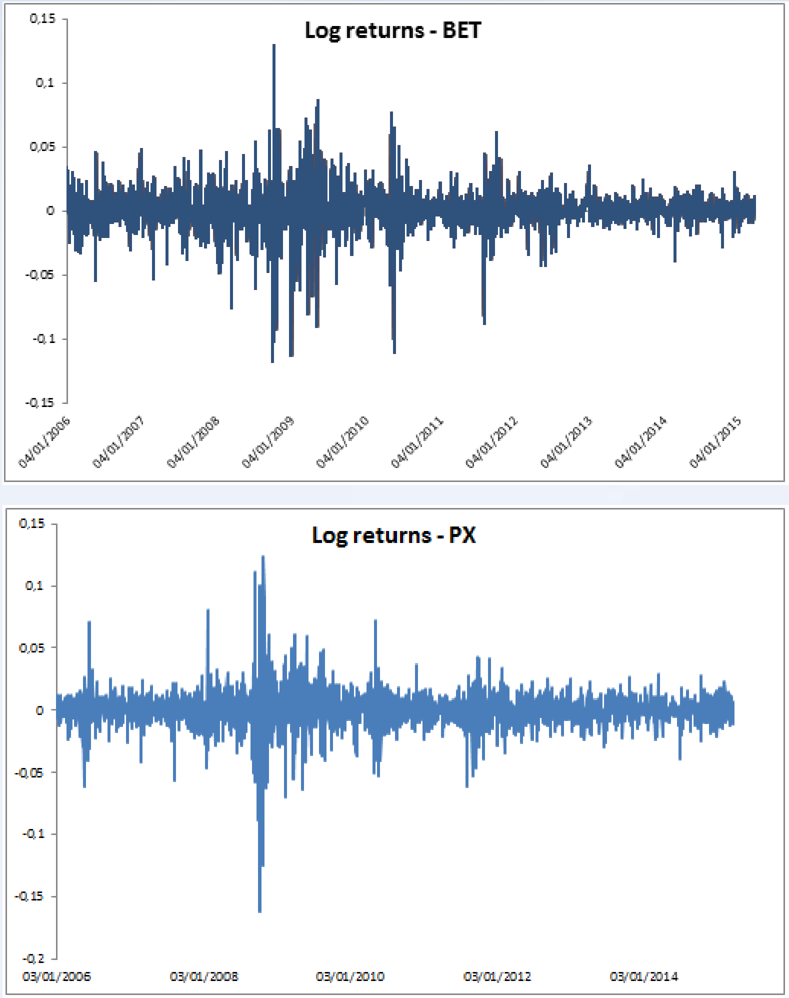
Index	Parameter	Mean	SD	5%	50%	95%
BET	μ	-8.942	0.1857	-9.2325	-8.943	-8.597
	Φ	0.965	0.0083	0.9452	0.965	0.972
	σ	0.317	0.0298	0.2778	0.317	0.373
	ρ	-0.2015	0.0758	-0.3378	-0.2015	-0.0463
	α_k	-0.0055	2,90E-05	-0.006	-0.0054	-0.004
	$\sqrt{\beta_k}$	0.031	6,60E-03	0.020	0.031	0.045
	κ	0.011	4,10E-03	0.003	0.0111	0.020
PX	μ	-9.061	0.1949	-9.3711	-9.061	-8.782
	Φ	0.979	0.0070	0.9672	0.979	0.989
	σ	0.218	0.0213	0.1797	0.217	0.253
	ρ	-0.2542	0.0823	-0.4014	-0.2541	-0.0722
	α_k	-0.0065	3,30E-05	-0.007	-0.0065	-0.006
	$\sqrt{\beta_k}$	0.036	7,20E-03	0.019	0.036	0.051
	κ	0.014	4,40E-03	0.004	0.0139	0.021
WIG20	μ	-8.802	0.3081	-9.3012	-8.801	-8.391
	Φ	0.989	0.0042	0.9829	0.990	0.997
	σ	0.121	0.0147	0.0983	0.121	0.147
	ρ	-0.2182	0.0818	-0.3469	-0.2181	-0.061
	α_k	-0.0074	3,50E-05	-0.008	-0.0074	-0.007
	$\sqrt{\beta_k}$	0.042	7,20E-03	0.024	0.042	0.052
	κ	0.017	4,60E-03	0.003	0.017	0.023
ATX	μ	-8.743	0.2612	-9.131	-8.743	-8.365
	Φ	0.986	0.0053	0.975	0.986	0.993
	σ	0.161	0.0185	0.131	0.160	0.193
	ρ	-0.2531	0.0812	-0.3817	-0.253	-0.065
	α	-0.0049	2,30E-05	-0.006	-0.0049	-0.004
	$\sqrt{\beta_k}$	0.029	5,60E-03	0.018	0.029	0.043
	κ	0.009	3,90E-03	0.002	0.009	0.018

*Estimated parameters – **SVLJ** model*
(Source: own computes)

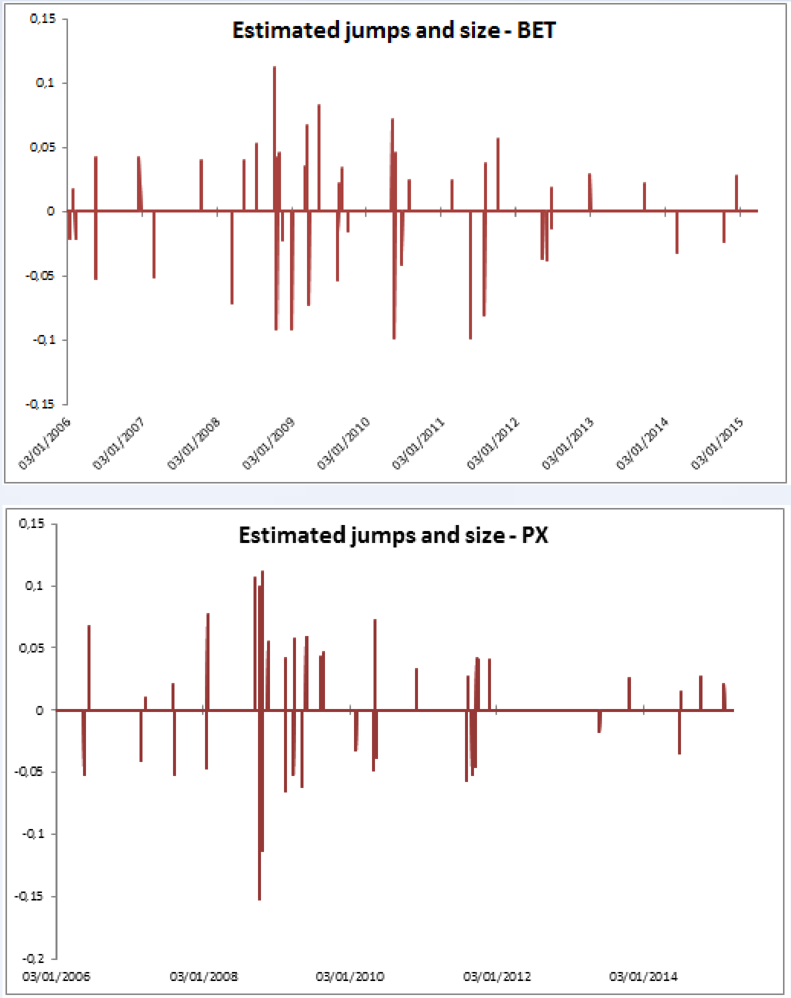


*Graphic of mean estimated volatilities – **SVLJ** model*
(Source: own computes)

Results and interpretations

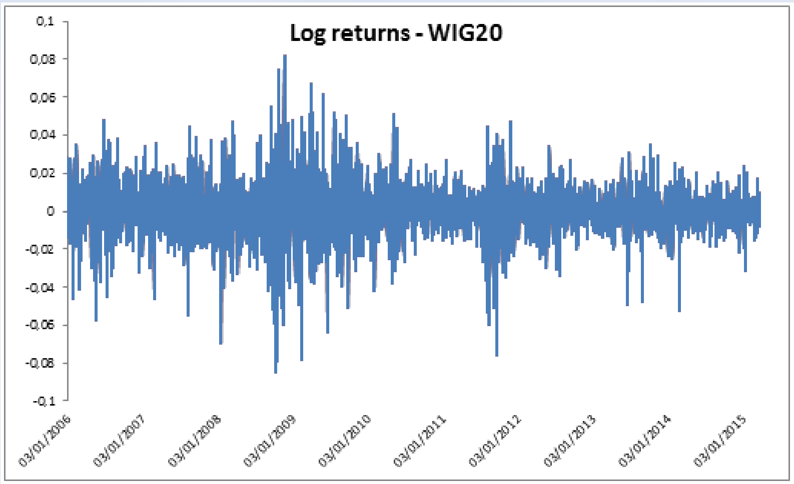


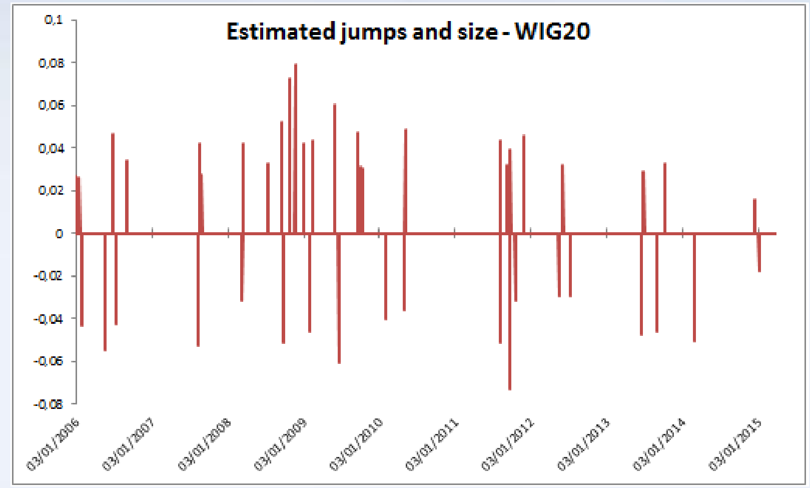
Log returns for BET and PX
(Source: own computes)

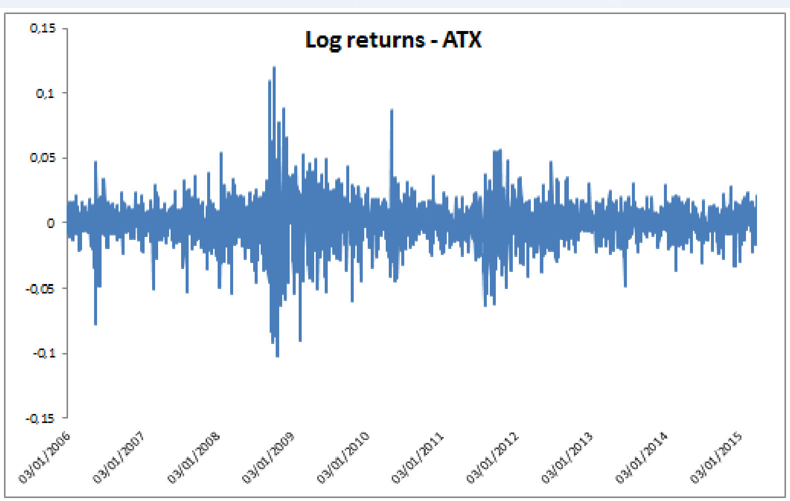


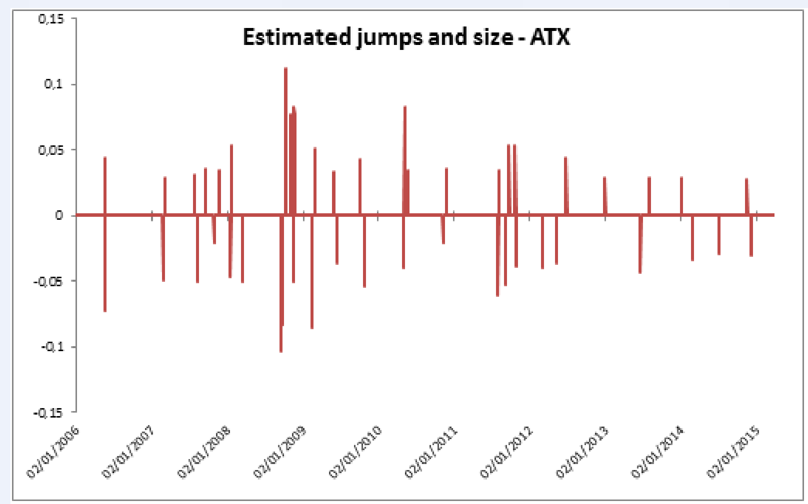
Estimated jumps and their size for BET and PX
(Source: own computes)

Results and interpretations





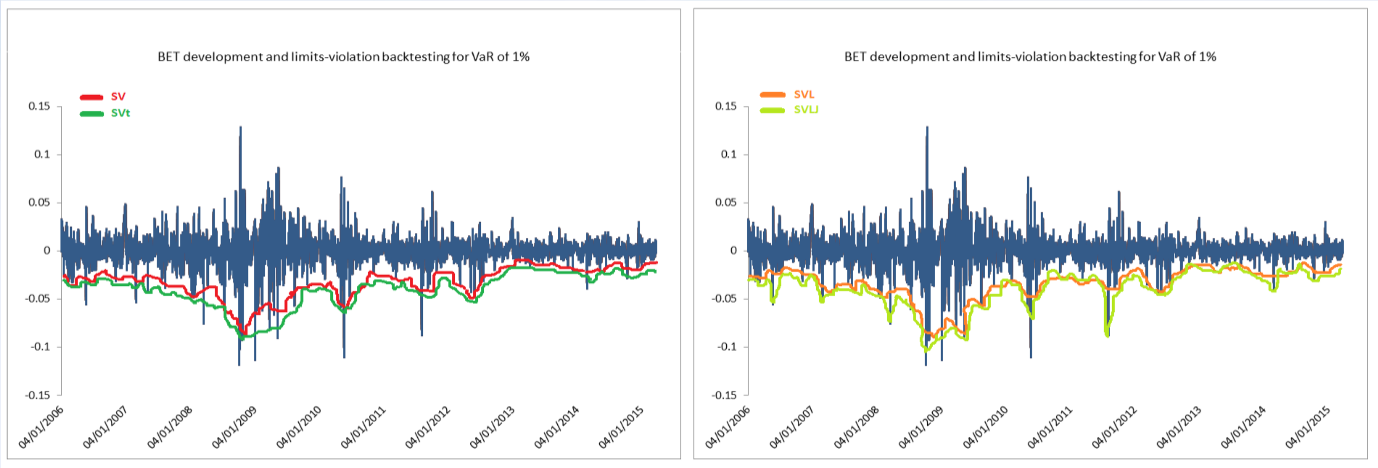




Log returns for BET and PX
(Source: own computes)

Estimated jumps and their size for BET and PX
(Source: own computes)

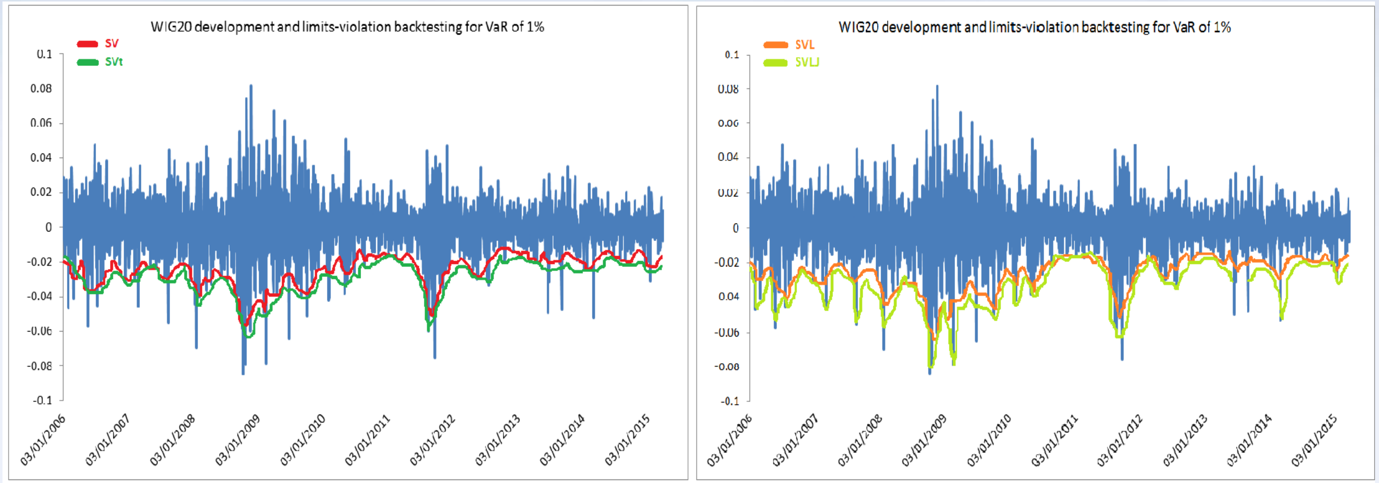
Results and interpretations



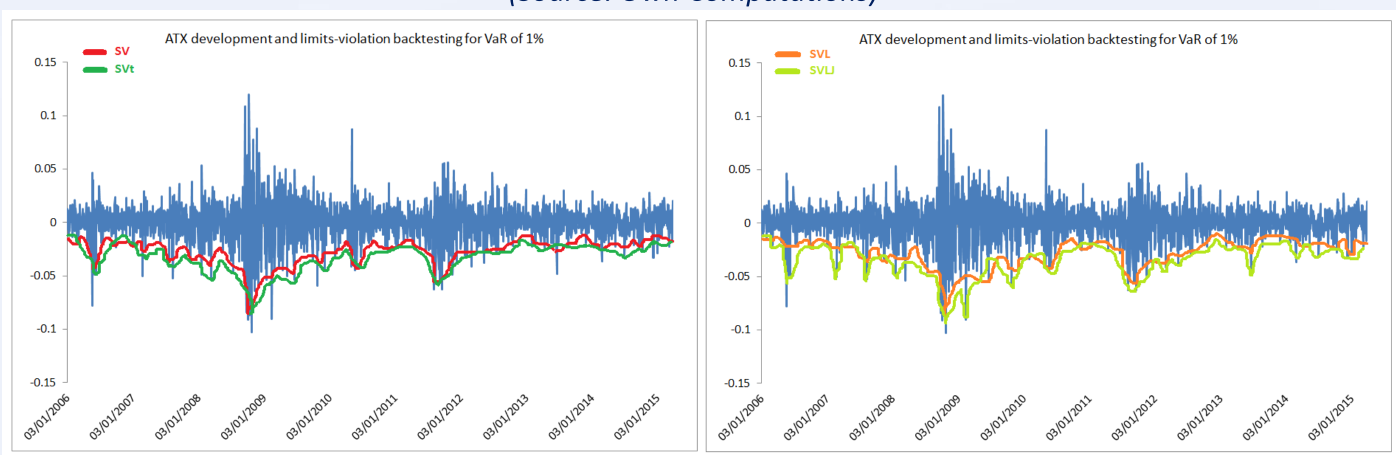
VaR 1% significance level, 1 day ahead volatility forecast, computation for BET – SV, SVt, SVL, SVLJ
(Source: Own Computations)

VaR 1% significance level, 1 day ahead volatility forecast, computation for PX – SV, SVt, SVL, SVLJ
(Source: Own Computations)

Results and interpretations



VaR 1% significance level, 1 day ahead volatility forecast, computation for WIG20 – SV, SVt, SVL, SVLJ (Source: Own Computations)



VaR 1% significance level, 1 day ahead volatility forecast, computation for ATX – SV, SVt, SVL, SVLJ (Source: Own Computations)

Results and interpretations

Index	Model	Limits-violations (% of total sample)
BET	SV	0.87%
	SVt	0.79%
	SVL	0.67%
	SVLJ	0.42%
PX	SV	0.92%
	SVt	0.81%
	SVL	0.72%
	SVLJ	0.45%
WIG20	SV	0.81%
	SVt	0.74%
	SVL	0.67%
	SVLJ	0.39%
ATX	SV	0.76%
	SVt	0.69%
	SVL	0.58%
	SVLJ	0.32%

Comparing method:
Testing the violation of the VaR limits given by the number of excesses outside the confidence interval.

VaR computation: 1% significance level, one-day-ahead volatility forecast.

Best results are obtained for the SVLJ model.

Results and interpretations

Index	Model	DIC
BET	SV	-11615.6
	SVt	-11725.3
	SVL	-11775.8
	SVLJ	-11975.7
PX	SV	-12248.8
	SVt	-12367.1
	SVL	-12512.9
	SVLJ	-12911.4
WIG20	SV	-13242.1
	SVt	-13312.4
	SVL	-13512.6
	SVLJ	-14121.7
ATX	SV	-11621.7
	SVt	-11748.1
	SVL	-11917.6
	SVLJ	-12218.3

(Source: Own Computations – R2OpenBugs)

y – the data, θ – the parameters

Deviance $D(\theta) = -2 * \log(p(y|\theta))$

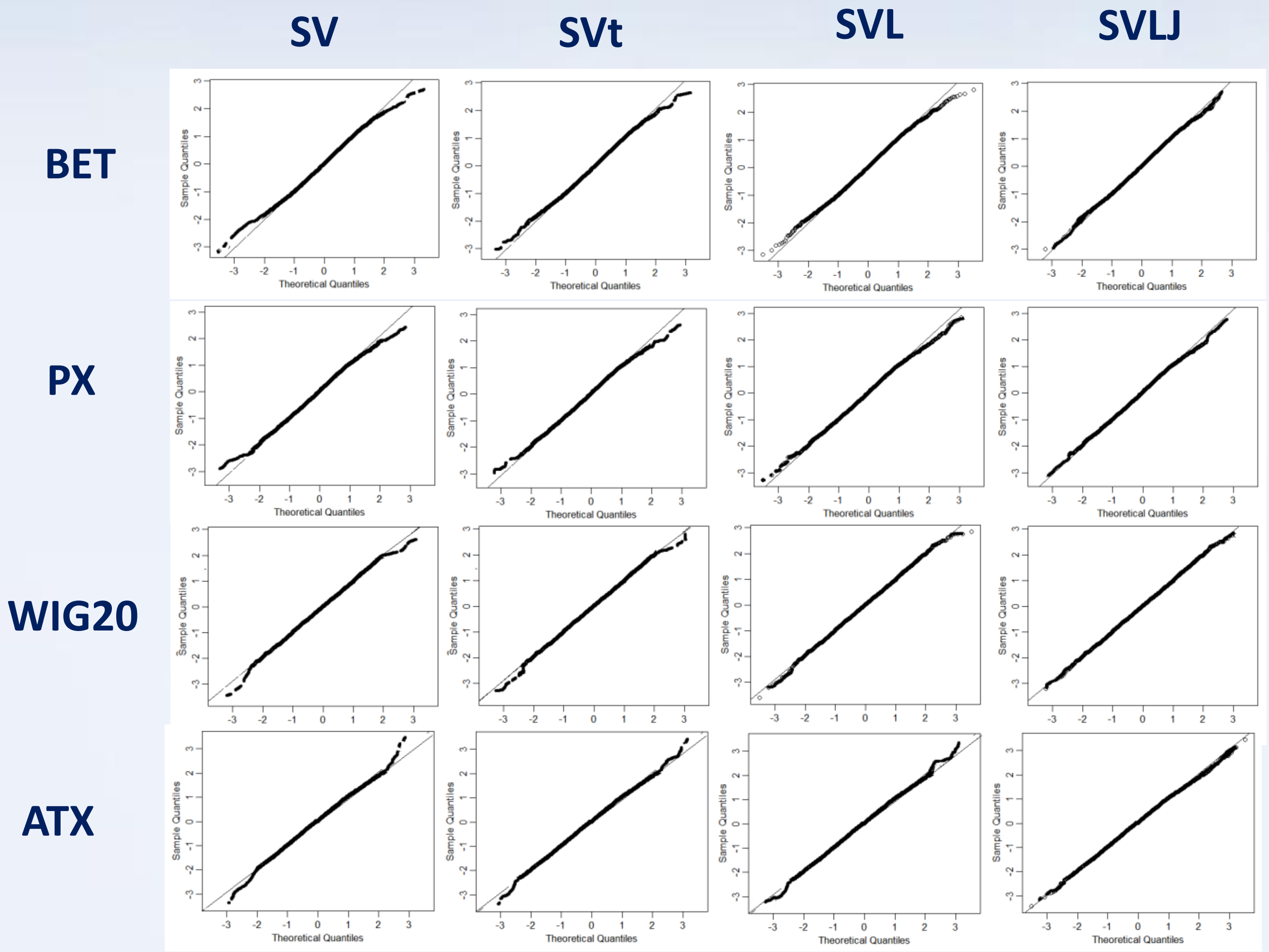
Expectation $\bar{D} = E[D(\theta)]$

Effective No. Params. $p_D = \bar{D} - D(\bar{\theta})$

Deviance Information Criterion

$$DIC = p_D + \bar{D}$$

Results and interpretations



Normal Q-Q for standardized residuals
(Source: Own Computations – R2OpenBugs)

Conclusions

The SVLJ model obtains the best performances in calibrating the data in comparison with SV, SVt, and SVL on all datasets:

- Smallest number of violations according to VaR;
- Best results according to DIC;
- Interpretation of Normal Q-Q for standardized residuals;
- Best representation of volatilities in turbulent periods.

Further research:

- considering models with other different particularities such as: markov regime switching stochastic volatility, models with jumps not only in returns but also in volatility; multivariate stochastic volatility models;
- applying the model for different datasets: stocks, exchange rates.

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